

INDUCED REPRESENTATIONS OF HILBERT MODULES OVER LOCALLY C*-ALGEBRAS AND THE IMPRIMITIVITY THEOREM

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ABSTRACT. We study induced representations of Hilbert modules over locally C*-algebras and their non-degeneracy. We show that if V and W are Morita equivalent Hilbert modules over locally C*-algebras A and B , respectively, then there exists a bijective correspondence between equivalence classes of non-degenerate representations of V and W .

1. INTRODUCTION

Morita equivalence and induced representations of C*-algebras were first introduced by Rieffel [16, 17]. Two C*-algebras A and B are Morita equivalent if there exists a full Hilbert A -module E such that B is isomorphic to the C*-algebra $K_A(E)$ of all compact operators on E . Some properties of C*-algebras that are preserved under Morita equivalence were investigated in [2, 4, 15, 21]. Indeed, Rieffel defined induced representations of C*-algebras, that are now known as Rieffel induced representations, by using tensor products of Hilbert modules and established an equivalence between the categories of non-degenerate representations of Morita equivalent C*-algebras. Joita [10, 11] defined the notions of Morita equivalence and induced representations in the category of locally C*-algebras. Joita and Moslehian [12] have recently introduced a notion of Morita equivalence in the category of Hilbert C*-modules considered to obtain induced representations of Hilbert modules over locally C*-algebras. This enables us to prove the imprimitivity theorem for induced representations of Hilbert modules over locally C*-algebras.

Let us quickly recall the definition of locally C*-algebras and Hilbert modules over them. A locally C*-algebra is a complete Hausdorff complex topological *-algebra A whose topology is determined by its continuous C*-seminorms in the sense that the net $\{a_i\}_{i \in I}$ converges to 0 if and only if the net $\{p(a_i)\}_{i \in I}$ converges to 0 for every continuous C*-seminorm p on A . Such algebras appear in the study of certain aspects of C*-algebras such as tangent algebras of C*-algebras, a domain of closed *-derivations on C*-algebras, multipliers of Pedersen's

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ideal, noncommutative analogues of classical Lie groups, and K-theory. These algebras were first introduced by Inoue [6] as a generalization of C*-algebras and studied more in [5, 14] with different names. A (right) *pre-Hilbert module* over a locally C*-algebra A is a right A -module E compatible with the complex algebra structure and equipped with an A -valued inner product $\langle \cdot, \cdot \rangle : E \times E \rightarrow A$, $(x, y) \mapsto \langle x, y \rangle$, which is A -linear in the second variable y and has the properties:

$$\langle x, y \rangle = \langle y, x \rangle^*, \text{ and } \langle x, x \rangle \geq 0 \text{ with equality if and only if } x = 0.$$

A pre-Hilbert A -module E is a Hilbert A -module if E is complete with respect to the topology determined by the family of seminorms $\{\bar{p}_E\}_{p \in S(A)}$, where $\bar{p}_E(\xi) = \sqrt{p(\langle \xi, \xi \rangle)}$, $\xi \in E$. Hilbert modules over locally C*-algebras have been studied systematically in the book [8] and the papers [7, 14, 20].

Joita and Moslehian [12], and Skeide [18] defined Morita equivalence for Hilbert C*-modules with two different methods. In the recent sense of Joita and Moslehian, two Hilbert modules V and W over C*-algebras A and B , respectively, are called Morita equivalent if $K_A(V)$ and $K_B(W)$ are strong Morita equivalent as C*-algebras. We consider this definition, which is weaker than Skeide's definition and also fitted to our paper.

In this paper, we first present some definitions and basic facts about locally C*-algebras and Hilbert modules over them. In [19], Skeide proved that if E is a Hilbert module over a C*-algebra A , then every representation of A induces a representation of E . We use this fact to reformulate the induced representations of Hilbert C*-modules and some of their properties which have been studied in [1]. These enable us to obtain the notion of induced representations of Hilbert modules over locally C*-algebras. We finally define the concept of Morita equivalence for Hilbert modules over locally C*-algebras. We prove that two full Hilbert modules over locally C*-algebras are Morita equivalent if and only if their underlying locally C*-algebras are strong Morita equivalent and then we give a module version of the imprimitivity theorem. Indeed, we show that for Morita equivalent Hilbert modules V and W over locally C*-algebras A and B , respectively, there is a bijective correspondence between equivalence classes of non-degenerate representations of V and W .

2. PRELIMINARIES

Let A be a locally C*-algebra, $S(A)$ the set of all continuous C*-seminorms on A and $p \in S(A)$. We set $N_p = \{a \in A : p(a) = 0\}$, then $A_p = A/N_p$ is a C*-algebra in the norm induced by p . For $p, q \in S(A)$ with $p \geq q$, the surjective morphisms $\pi_{pq} : A_p \rightarrow A_q$ defined by $\pi_{pq}(a + N_p) = a + N_q$ induce the inverse system $\{A_p; \pi_{pq}\}_{p, q \in S(A), p \geq q}$ of C*-algebras and

$A = \varprojlim_p A_p$, i.e., the locally C^* -algebra A can be identified with $\varprojlim_p A_p$. The canonical map from A onto A_p is denoted by π_p and a_p is reserved to denote $a + N_p$. A morphism of locally C^* -algebras is a continuous morphism of $*$ -algebras. An isomorphism of locally C^* -algebras is a morphism of locally C^* -algebras which possesses an inverse morphism of locally C^* -algebras.

A representation of a locally C^* -algebra A is a continuous $*$ -morphism $\varphi : A \rightarrow B(H)$, where $B(H)$ is the C^* -algebra of all bounded linear maps on a Hilbert space H . If (φ, H) is a representation of A , then there is $p \in S(A)$ such that $\|\varphi(a)\| \leq p(a)$, for all $a \in A$. The representation (φ_p, H) of A_p , where $\varphi_p \circ \pi_p = \varphi$ is called a representation of A_p associated to (φ, H) . We refer to [5, 11] for basic facts and definitions about the representation of locally C^* -algebras.

Suppose E is a Hilbert A -module and $\langle E, E \rangle$ is the closure of linear span of $\{\langle x, y \rangle : x, y \in E\}$. The Hilbert A -module E is called *full* if $\langle E, E \rangle = A$. One can always consider any Hilbert A -module as a full Hilbert module over locally C^* -algebra $\langle E, E \rangle$. For each $p \in S(A)$, $N_p^E = \{\xi \in E : \bar{p}_E(\xi) = 0\}$ is a closed submodule of E and $E_p = E/N_p^E$ is a Hilbert A_p -module with the action $(\xi + N_p^E)\pi_p(a) = \xi a + N_p^E$ and the inner product $\langle \xi + N_p^E, \eta + N_p^E \rangle = \pi_p(\langle \xi, \eta \rangle)$. The canonical map from E onto E_p is denoted by σ_p^E and ξ_p is reserved to denote $\sigma_p^E(\xi)$. For $p, q \in S(A)$ with $p \geq q$, the surjective morphisms $\sigma_{pq}^E : E_p \rightarrow E_q$ defined by $\sigma_{pq}^E(\sigma_p^E(\xi)) = \sigma_q^E(\xi)$ induce the inverse system $\{E_p; A_p; \sigma_{pq}^E, \pi_{pq}\}_{p, q \in S(A), p \geq q}$ of Hilbert C^* -modules in the following sense:

- $\sigma_{pq}^E(\xi_p a_p) = \sigma_{pq}^E(\xi_p) \pi_{pq}(a_p)$, $\xi_p \in E_p$, $a_p \in A_p$, $p, q \in S(A)$, $p \geq q$,
- $\langle \sigma_{pq}^E(\xi_p), \sigma_{pq}^E(\eta_p) \rangle = \pi_{pq}(\langle \xi_p, \eta_p \rangle)$, $\xi_p, \eta_p \in E_p$, $p, q \in S(A)$, $p \geq q$,
- $\sigma_{qr}^E \circ \sigma_{pq}^E = \sigma_{pr}^E$ if $p, q, r \in S(A)$ and $p \geq q \geq r$,
- $\sigma_{pp}^E(\xi_p) = \xi_p$, $\xi \in E$, $p \in S(A)$.

In this case, $\varprojlim_p E_p$ is a Hilbert A -module which can be identified with E . Let E and F be Hilbert A -modules and $T : E \rightarrow F$ an A -module map. The module map T is called bounded if for each $p \in S(A)$ there is $k_p > 0$ such that $\bar{p}_F(Tx) \leq k_p \bar{p}_E(x)$ for all $x \in E$. The module map T is called adjointable if there exists an A -module map $T^* : F \rightarrow E$ with the property $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x \in E, y \in F$. It is well-known that every adjointable map is bounded. The set $L_A(E, F)$ of all bounded adjointable A -module maps from E into F becomes a locally convex space with the topology defined by the family of seminorms $\{\tilde{p}\}_{p \in S(A)}$, where $\tilde{p}(T) = \|(\pi_p)_*(T)\|_{L_{A_p}(E_p, F_p)}$ and $(\pi_p)_* : L_A(E, F) \rightarrow L_{A_p}(E_p, F_p)$ is defined by $(\pi_p)_*(T)(\xi + N_p^E) = T\xi + N_p^F$ for all $T \in L_A(E, F)$, $\xi \in E$. For $p, q \in S(A)$ with $p \geq q$, the morphisms $(\pi_{pq})_* : L_{A_p}(E_p, F_p) \rightarrow L_{A_q}(E_q, F_q)$ defined by $(\pi_{pq})_*(T_p)(\sigma_q^E(\xi)) =$

$\sigma_{pq}^F(T_p(\sigma_p^E(\xi)))$ induce the inverse system

$$\{L_{A_p}(E_p, F_p); (\pi_{pq})_*\}_{p,q \in S(A), p \geq q}$$

of Banach spaces such that $\varprojlim_p L_{A_p}(E_p, F_p)$ can be identified to $L_A(E, F)$. In particular, topologizing, $L_A(E, E)$ becomes a locally C^* -algebra which is abbreviated by $L_A(E)$. The set of all compact operators $K_A(E)$ on E is defined as the closed linear subspace of $L_A(E)$ spanned by $\{\theta_{x,y} : \theta_{x,y}(\xi) = x\langle y, \xi \rangle \text{ for all } x, y, \xi \in E\}$. This is a locally C^* -subalgebra and a two-sided ideal of $L_A(E)$; moreover, $K_A(E)$ can be identified to $\varprojlim_p K_{A_p}(E_p)$.

Let V and W be Hilbert modules over locally C^* -algebras A and B , respectively, and $\Psi : A \rightarrow L_B(W)$ a continuous $*$ -morphism. We can regard W as a left A -module by $(a, y) \rightarrow \Psi(a)y$, $a \in A$, $y \in W$. The right B -module $V \otimes_A W$ is a pre-Hilbert module with the inner product given by $\langle x \otimes y, z \otimes t \rangle = \langle y, \Psi(\langle x, z \rangle)t \rangle$. We denote by $V \otimes_\Psi W$ the completion of $V \otimes_A W$, cf. [9] for more detailed information.

3. INDUCED REPRESENTATIONS OF HILBERT MODULES

In this section, we first study induced representations of Hilbert C^* -modules and then we reformulate them in the context of Hilbert modules over locally C^* -algebras.

Let H and K be Hilbert spaces. Then the space $B(H, K)$ of all bounded operators from H into K can be considered as a Hilbert $B(H)$ -module with the module action $(T, S) \rightarrow TS$, $T \in B(H, K)$ and $S \in B(H)$ and the inner product defined by $\langle T, S \rangle = T^*S$, $T, S \in B(H, K)$. Murphy [13] showed that any Hilbert C^* -module can be represented as a submodule of the concrete Hilbert module $B(H, K)$ for some Hilbert spaces H and K . This allows us to extend the notion of a representation from the context of C^* -algebras to the context of Hilbert C^* -modules. Let V and W be two Hilbert modules over C^* -algebras A and B , respectively, and $\varphi : A \rightarrow B$ be a morphism of C^* -algebras. A map $\Phi : V \rightarrow W$ is said to be a φ -morphism if $\langle \Phi(x), \Phi(y) \rangle = \varphi(\langle x, y \rangle)$ for all $x, y \in V$. A φ -morphism $\Phi : V \rightarrow B(H, K)$, where $\varphi : A \rightarrow B(H)$ is a representation of A is called a representation of V . When Φ is a representation of V , we assume that an associated representation of A is denoted by the same lowercase letter φ , so we will not explicitly mention φ . Let $\Phi : V \rightarrow B(H, K)$ be a representation of a Hilbert A -module V . We say Φ is a non-degenerate representation if $\overline{\Phi(V)(H)} = K$ and $\overline{\Phi(V)^*(K)} = H$. Two representations $\Phi_i : V \rightarrow B(H_i, K_i)$ of V , $i = 1, 2$ are said to be unitarily equivalent if there are unitary operators $U_1 : H_1 \rightarrow H_2$ and $U_2 : K_1 \rightarrow K_2$, such that $U_2\Phi_1(v) = \Phi_2(v)U_1$ for all $v \in V$. Representations of Hilbert modules have been investigated in [1, 3, 19].

Lemma 3.1. *Let V be a full Hilbert A -module and $\Phi_1 : V \rightarrow B(H_1, K_1)$ and $\Phi_2 : V \rightarrow B(H_2, K_2)$ two non-degenerate representations of V . If Φ_1 and Φ_2 are unitarily equivalent, then φ_1 and φ_2 are unitarily equivalent.*

Proof. Let $U_1 : H_1 \rightarrow H_2$ and $U_2 : K_1 \rightarrow K_2$ be unitary operators and $U_2\Phi_1(x) = \Phi_2(x)U_1$ for all $x \in V$. Then we have

$$U_1\varphi_1(\langle x, y \rangle)h = U_1\Phi_1(x)^*\Phi_1(y)h = \Phi_2(x)^*\Phi_2(y)U_1h = \varphi_2(\langle x, y \rangle)U_1h,$$

for every $x, y \in V$ and $h \in H_1$. Since V is full, we conclude that $U_1\varphi_1(a)h = \varphi_2(a)U_1h$ for every $a \in A$ and $h \in H_1$, and consequently, φ_1 and φ_2 are unitarily equivalent. \square

Skeide [19] recovered the result of Murphy by embedding every Hilbert A -module E into a matrix C^* -algebra as a lower submodule. He proved that every representation of B induces a representation of E . We describe his induced representation as follows.

Construction 3.2. Let B be a C^* -algebra and E a Hilbert B -module and $\varphi : B \rightarrow B(H)$ a $*$ -representation of B . Define a sesquilinear form $\langle \cdot, \cdot \rangle$ on the vector space $E \otimes_{alg} H$ by $\langle x \otimes h, y \otimes k \rangle = \langle h, \varphi(\langle x, y \rangle)k \rangle_H$, where $\langle \cdot, \cdot \rangle_H$ denotes the inner product on the Hilbert space H . By [19, Proposition 3.8], the sesquilinear form is positive and so $E \otimes_{alg} H$ is a semi-Hilbert space. Then $(E \otimes_{alg} H)/N_\varphi$ is a pre-Hilbert space with the inner product defined by

$$\langle x \otimes h + N_\varphi, y \otimes k + N_\varphi \rangle = \langle x \otimes h, y \otimes k \rangle,$$

where N_φ is the vector subspace of $E \otimes_{alg} H$ generated by $\{x \otimes h \in E \otimes_{alg} H : \langle x \otimes h, x \otimes h \rangle = 0\}$. The completion of $(E \otimes_{alg} H)/N_\varphi$ with respect to the above inner product is denoted by ${}_E H$. We identify the elements $x \otimes h$ with the equivalence classes $x \otimes h + N_\varphi \in {}_E H$. Suppose $x \in E$ and $L_x h = x \otimes h$ then $\|L_x h\|^2 = \langle h, \varphi(\langle x, x \rangle)h \rangle \leq \|h\|^2 \|x\|^2$, i.e. $L_x \in B(H, {}_E H)$. We define $\eta_\varphi : E \rightarrow B(H, {}_E H)$ by $\eta_\varphi(x) = L_x$. Then for $x, x' \in E$, $h, h' \in H$ and $b \in B$ we have $\langle \eta_\varphi(x), \eta_\varphi(x') \rangle = \varphi(\langle x, x' \rangle)$ and $\eta_\varphi(xb) = \eta_\varphi(x)\varphi(b)$, and so η_φ is a representation of E .

Lemma 3.3. *Let $\varphi_1 : B \rightarrow B(H_1)$ and $\varphi_2 : B \rightarrow B(H_2)$ be two non-degenerate representations of B . If φ_1 and φ_2 are unitarily equivalent, then η_{φ_1} and η_{φ_2} are unitarily equivalent.*

Proof. Suppose $U : H_1 \rightarrow H_2$ is a unitary operator such that $U\varphi_1(b) = \varphi_2(b)U$ for all $b \in B$. Then $id_E \otimes U : E \otimes_{alg} H_1 \rightarrow E \otimes_{alg} H_2$ given by $x \otimes h_1 \mapsto x \otimes h_2$ can be extended to a unitary operator V from ${}_E H_1$ onto ${}_E H_2$ and $V\eta_{\varphi_1}(x) = \eta_{\varphi_2}(x)U$ for all $x \in E$. Hence, η_{φ_1} and η_{φ_2} are unitarily equivalent. \square

The above argument enables us to extend the Rieffel induced representations from the case of C^* -algebras to the context of Hilbert C^* -modules. For this, let V and W be two full

Hilbert modules over C^* -algebras A and B , respectively. Let E be a Hilbert B -module and A acts as adjointable operators on the Hilbert C^* -module E , and $\Phi : W \rightarrow B(H, K)$ is a non-degenerate representation of W . Using [15, Proposition 2.66], the formula ${}^A_E\varphi(x \otimes h) = (a.x) \otimes h$ extends to obtain a (Rieffel induced) representation of A as bounded operators on Hilbert space ${}_E H$. In view of Construction 3.2, the representation ${}^A_E\varphi : A \rightarrow B({}_E H)$ of the C^* -algebra A obtains the representation $\eta_{{}^A_E\varphi} : V \rightarrow B({}_E H, {}_V({}_E H))$ of the Hilbert A -module V . The constructed representation $\eta_{{}^A_E\varphi}$ is called the *Rieffel induced representation* from W to V via E and denoted by ${}_E^V\Phi$. The following result can be found in [1, Proposition 3.3] that we derive from Lemmas 3.1 and 3.3. Our argument seems to be shorter.

Lemma 3.4. *Let W be a full Hilbert B -module and $\Phi_1 : W \rightarrow B(H_1, K_1)$ and $\Phi_2 : W \rightarrow B(H_2, K_2)$ two non-degenerate representations of W . If Φ_1 and Φ_2 are unitarily equivalent, then ${}_E^V\Phi_1$ and ${}_E^V\Phi_2$ are unitarily equivalent.*

Corollary 3.5. *If $\Phi : W \rightarrow B(H, K)$ and $\oplus_{i \in I} \Phi_i : W \rightarrow B(\oplus_{i \in I} H_i, \oplus_{i \in I} K_i)$ are unitarily equivalent, then ${}_E^V\Phi$ and $\oplus_{i \in I} {}_E^V\Phi_i$ are unitarily equivalent.*

Now, we reformulate representations of the Hilbert module from the case of C^* -algebras to the case of locally C^* -algebras. Let V and W be two Hilbert modules over locally C^* -algebras A and B , respectively, and $\varphi : A \rightarrow B$ a morphism of locally C^* -algebras. A map $\Phi : V \rightarrow W$ is said to be a φ -morphism if $\langle \Phi(x), \Phi(y) \rangle = \varphi(\langle x, y \rangle)$, for all $x, y \in V$. A φ -morphism $\Phi : V \rightarrow B(H, K)$, where $\varphi : A \rightarrow B(H)$ is a representation of A , is called a representation of V . We can define non-degenerate representations and unitarily equivalent representations for Hilbert modules over locally C^* -algebras like a Hilbert C^* -modules case.

Suppose A is a locally C^* -algebra, V is a Hilbert A -module and $\varphi : A \rightarrow B(H)$ is a representation of A on some Hilbert space H . Suppose $p \in S(A)$ and φ_p is a representation of A_p associated to φ ; then there exist a Hilbert space K and a representation $\Phi_p : V_p \rightarrow B(H, K)$ which is a φ_p -morphism. For details we refer to the proof of [13, Theorem 3.1]. It is easy to see that the map $\Phi : V \rightarrow B(H, K)$, $\Phi(v) = \Phi_p(\sigma_p^V(v))$ is a φ -morphism, i.e., it is a representation of V .

Lemma 3.6. *Let V be a Hilbert module over locally C^* -algebra A and $\Phi : V \rightarrow B(H, K)$ a representation of V . If $p \in S(A)$ and φ_p is a representation of A_p associated to φ , then the map $\Phi_p : V_p \rightarrow B(H, K)$, $\Phi_p(\sigma_p^V(v)) = \Phi(v)$ is a φ_p -morphism. Specifically, Φ_p is a representation of V_p and Φ is non-degenerate if and only if Φ_p is. In this case, we say that Φ_p is a representation of V_p associated to Φ .*

Proof. Let $v, v' \in V$ and $\bar{p}_V(v - v') = 0$. Since $\|\varphi(a)\| \leq p(a)$ for all $a \in A$, we have $\langle \Phi(v - v'), \Phi(v - v') \rangle = \varphi(\langle v - v', v - v' \rangle) = 0$, which shows Φ_p is well-defined. We also have

$$\begin{aligned} \langle \Phi_p(\sigma_p^V(v)), \Phi_p(\sigma_p^V(v')) \rangle &= \langle \Phi(v), \Phi(v') \rangle = \varphi(\langle v, v' \rangle) = \varphi_p \circ \pi_p(\langle v, v' \rangle) \\ &= \varphi_p(\langle \sigma_p^V(v), \sigma_p^V(v') \rangle). \end{aligned}$$

Then, by definition of Φ_p , the representation Φ is non-degenerate if and only if Φ_p is non-degenerate. \square

Let V and W be two full Hilbert modules over locally C^* -algebras A and B , respectively. Let E be a Hilbert B -module, $\Psi : A \rightarrow L_B(E)$ a non-degenerate continuous $*$ -morphism and $\Phi : W \rightarrow B(H, K)$ a non-degenerate representation of W . We construct a non-degenerate representation from W to V via E as follows.

Construction 3.7. We define a sesquilinear form $\langle \cdot, \cdot \rangle$ on the vector space $E \otimes_{alg} H$ by $\langle x \otimes h, y \otimes k \rangle = \langle h, \varphi(\langle x, y \rangle)k \rangle_H$ and make the Hilbert space ${}_E H$ as in Construction 3.2. The map ${}_E^A \varphi : A \rightarrow B({}_E H)$ defined by

$${}_E^A \varphi(a)(x \otimes h) = \Psi(a)x \otimes h, \quad a \in A, \quad x \in E, \quad h \in H,$$

is a representation of A . The representation $({}_E H, {}_E^A \varphi)$ is called the *Rieffel induced representation* from B to A via E , cf. [11]. Since A acts as an adjointable operator on Hilbert B -module E , we can construct interior tensor product $V \otimes_\Psi E$ as a Hilbert B -module. Hence, we find the Hilbert spaces ${}_E H$ and ${}_{V \otimes_\Psi E} H$. Let $v \in V$; then the map $E \times H \rightarrow {}_{V \otimes_\Psi E} H$, $(x, h) \mapsto v \otimes x \otimes h$ is a bilinear form and so there is a unique linear transformation ${}_E \Phi(v) : E \otimes_{alg} H \rightarrow {}_{V \otimes_\Psi E} H$ which can be extended to a bounded linear operator ${}_E^V \Phi(v)$ from ${}_E H$ to ${}_{V \otimes_\Psi E} H$. To see this, suppose $q \in S(B)$, $x \in E$, $h \in H$ and (φ_q, H) is a representation of B_q associated to (φ, H) . We have

$$\begin{aligned}
\langle {}_E\Phi(v)(x \otimes h) \rangle, \quad {}_E\Phi(v)(x \otimes h) \rangle &= \langle v \otimes x \otimes h, v \otimes x \otimes h \rangle \\
&= \langle h, \varphi(\langle v \otimes x, v \otimes x \rangle)h \rangle_H \\
&= \langle h, \varphi(\langle x, \Psi(\langle v, v \rangle)x \rangle)h \rangle_H \\
&= \langle h, \varphi_q \circ \pi_q(\langle \Psi(\langle v, v \rangle)^{1/2}x, \Psi(\langle v, v \rangle)^{1/2}x \rangle)h \rangle_H \\
&= \langle h, \varphi_q(\langle \sigma_q(\Psi(\langle v, v \rangle)^{1/2}x), \sigma_q(\Psi(\langle v, v \rangle)^{1/2}x) \rangle)h \rangle_H \\
&= \langle h, \varphi_q(\langle (\pi_q)_*(\Psi(\langle v, v \rangle)^{1/2})(\sigma_q(x)), (\pi_q)_*(\Psi(\langle v, v \rangle)^{1/2})(\sigma_q(x)) \rangle)h \rangle_H \\
&\leq \tilde{q}(\Psi\langle v, v \rangle)\langle h, \varphi_q(\langle \sigma_q(x), \sigma_q(x) \rangle)h \rangle_H \\
&= \tilde{q}(\Psi\langle v, v \rangle)\langle h, (\varphi_q \circ \pi_q)(\langle x, x \rangle)h \rangle_H \\
&= \tilde{q}(\Psi\langle v, v \rangle)\langle h, \varphi(\langle x, x \rangle)h \rangle_H \\
&= \tilde{q}(\Psi\langle v, v \rangle)\langle x \otimes h, x \otimes h \rangle.
\end{aligned}$$

The following equalities hold for every $v, v' \in V$, $x, x' \in E$ and $h, h' \in H$

$$\begin{aligned}
\langle x \otimes h, {}^V_E\Phi^*(v) {}^V_E\Phi(v')(x' \otimes h') \rangle &= \langle {}^V_E\Phi(v)(x \otimes h), {}^V_E\Phi(v')(x' \otimes h') \rangle \\
&= \langle v \otimes x \otimes h, v' \otimes x' \otimes h' \rangle \\
&= \langle h, \varphi(\langle v \otimes x, v' \otimes x' \rangle)h \rangle_H \\
&= \langle h, \varphi(\langle x, \Psi(\langle v, v' \rangle)x' \rangle)h' \rangle_H \\
&= \langle x \otimes h, \Psi(\langle v, v' \rangle)x' \otimes h' \rangle \\
&= \langle x \otimes h, {}^A_E\varphi(\langle v, v' \rangle)(x' \otimes h') \rangle,
\end{aligned}$$

which imply $\langle {}^V_E\Phi(v), {}^V_E\Phi(v') \rangle = {}^V_E\Phi^*(v) {}^V_E\Phi(v') = {}^A_E\varphi(\langle v, v' \rangle)$. That is, the map ${}^V_E\Phi : V \rightarrow B({}_E H, {}_{V \otimes {}_\Psi E} H)$ is a ${}^A_E\varphi$ -morphism and so it is a representation of V . We now show that ${}^V_E\Phi$ is non-degenerate. To see this, recall that $\overline{\Psi(A)(E)} = E$ and $\overline{\langle V, V \rangle} = A$, which imply $\overline{\Psi(\langle V, V \rangle)(E)} = E$. Suppose $x, x' \in E$ and $h \in H$, we have

$$\begin{aligned}
\|(x - x') \otimes h\|^2 &= \langle h, \varphi(\langle x - x', x - x' \rangle)h \rangle_H \\
&\leq \|h\|^2 \|\varphi(\langle x - x', x - x' \rangle)\| \\
&\leq \|h\|^2 q(\langle x - x', x - x' \rangle) = \|h\|^2 \bar{q}_E(x - x').
\end{aligned}$$

Given $\epsilon > 0$, there exist $v_i, v'_i \in V$ and $x_i \in E$ such that $\bar{q}_E(\sum_i \Psi(\langle v_i, v'_i \rangle)x_i - x) < \epsilon$. In view of the above inequality, the term $\sum_i \Psi(\langle v_i, v'_i \rangle)x_i \otimes h$ approximates $x \otimes h$ in ${}_E H$. But

we have

$$\begin{aligned}
\sum_i \Psi(\langle v_i, v'_i \rangle) x_i \otimes h &= \sum_i {}^A_E \varphi(\langle v_i, v'_i \rangle)(x_i \otimes h) \\
&= \sum_i {}^V_E \Phi^*(v_i) {}^V_E \Phi(v'_i)(x_i \otimes h) \\
&= \sum_i {}^V_E \Phi^*(v_i)(v'_i \otimes x_i \otimes h),
\end{aligned}$$

which implies ${}^V_E \Phi(V)^*({}_{V \otimes {}^\Psi E} H) = {}_E H$. The equality ${}^V_E \Phi(V)({}_E H) = {}_{V \otimes {}^\Psi E} H$ follows from the definition of ${}^V_E \Phi$, i.e., ${}^V_E \Phi$ is non-degenerate.

Definition 3.8. The representation ${}^V_E \Phi$ in Construction 3.7 is called Rieffel induced representation from W to V via E .

Theorem 3.9. Let V and W be two full Hilbert modules over locally C^* -algebras A and B , respectively. Let E be a Hilbert B -module, $\Psi : A \rightarrow L_B(E)$ a non-degenerate continuous $*$ -morphism and $\Phi : W \rightarrow B(H, K)$ a non-degenerate representation. If $q \in S(B)$ and (φ_q, H) is a non-degenerate representation of B_q associated to (φ, H) , then there is $p \in S(A)$ such that A_p acts non-degenerately on E_q and the representations ${}^V_E \Phi$ and ${}^{V_p}_{E_q} \Phi_q \circ \sigma_p^V$ of V are unitarily equivalent.

Proof. Continuity of Ψ implies that there exists $p \in S(A)$ such that $\tilde{q}(\Psi(a)) \leq p(a)$ for each $a \in A$, which guarantees $\Psi_p : A_p \rightarrow L_{B_q}(E_q)$, $\Psi_p(\pi_p(a)) = (\pi_q)_*(\Psi(a))$ is a $*$ -morphism of C^* -algebras. Moreover, Ψ_p is non-degenerate since

$$\begin{aligned}
\overline{\Psi_p(A_p)(E_p)} &= \overline{\Psi_p(\pi_p(A))(\sigma_p^E(E))} = \overline{(\pi_q)_*(\Psi(A)\sigma_q^E(E))} \\
&= \sigma_q^E(\overline{\Psi(A)(E)}) \\
&= \sigma_q^E(E) = E_q.
\end{aligned}$$

If Φ_q is a non-degenerate representation of W_q associated to Φ , then ${}^{V_p}_{E_q} \Phi_q : V_p \rightarrow B({}_E H, {}_{V_p \otimes {}^\Psi E_q} H)$ defined by ${}^{V_p}_{E_q} \Phi_q(\sigma_p^V(v))(\sigma_q^E(x) \otimes h) = \sigma_p^V(v) \otimes \sigma_q^E(x) \otimes h$ is a non-degenerate representation of V_p which is also a ${}^{A_p}_{E_q} \varphi_q$ -morphism. Indeed, ${}^{V_p}_{E_q} \Phi_q$ is the Rieffel induced representation from W_q to V_p via E_q . Hence, ${}^{V_p}_{E_q} \Phi_q \circ \sigma_p^V$ is a non-degenerate representation of V and it is a ${}^{A_p}_{E_q} \varphi_q \circ \pi_p$ -morphism. The representations $({}^A_E \varphi, {}_E H)$ and $({}^{A_p}_{E_q} \varphi_q \circ \pi_p, {}_E H)$ of A are unitarily equivalent by [11, proposition 3.4]. We define the linear map $U_1 : E \otimes_{alg} H \rightarrow E_q \otimes_{alg} H$,

$U_1(x \otimes h) = \sigma_q^E(x) \otimes h$ which satisfies

$$\begin{aligned}
\langle U_1(x \otimes h), U_1(x \otimes h) \rangle &= \langle \sigma_q^E(x) \otimes h, \sigma_q^E(x) \otimes h \rangle \\
&= \langle h, \varphi_q(\langle \sigma_q^E(x), \sigma_q^E(x) \rangle) h \rangle_H \\
&= \langle h, \varphi_q(\pi_q(\langle x, x \rangle)) h \rangle_H \\
&= \langle h, \varphi(\langle x, x \rangle) h \rangle_H \\
&= \langle x \otimes h, x \otimes h \rangle,
\end{aligned}$$

for all $x \in E$ and $h \in H$. Then U_1 can be extended to a bounded linear operator, which is again denoted by U_1 from ${}_E H$ onto ${}_{E_q} H$. It is easy to see that U_1 is a unitary operator. We define the linear map $U_2 : V \otimes_{alg} E \otimes_{alg} H \rightarrow V_p \otimes_{alg} E_q \otimes_{alg} H$ by $U_2(v \otimes x \otimes h) = \sigma_p^V(v) \otimes \sigma_q^E(x) \otimes h$. For every $v \in V$, $x \in E$ and $h \in H$ we have

$$\begin{aligned}
\langle U_2(v \otimes x \otimes h), U_2(v \otimes x \otimes h) \rangle &= \langle \sigma_p^V(v) \otimes \sigma_q^E(x) \otimes h, \sigma_p^V(v) \otimes \sigma_q^E(x) \otimes h \rangle \\
&= \langle h, \varphi_q(\langle \sigma_p^V(v) \otimes \sigma_q^E(x), \sigma_p^V(v) \otimes \sigma_q^E(x) \rangle) h \rangle_H \\
&= \langle h, \varphi_q(\langle \sigma_q^E(x), \Psi_p(\langle \sigma_p^V(v), \sigma_p^V(v) \rangle) \sigma_q^E(x) \rangle) h \rangle_H \\
&= \langle h, \varphi_q(\langle \sigma_q^E(x), \Psi_p(\pi_p(\langle v, v \rangle)) \sigma_q^E(x) \rangle) h \rangle_H \\
\\
\langle U_2(v \otimes x \otimes h), U_2(v \otimes x \otimes h) \rangle &= \langle h, \varphi_q(\langle \sigma_q^E(x), (\pi_q)_*(\Psi(\langle v, v \rangle)) \sigma_q^E(x) \rangle) h \rangle_H \\
&= \langle h, \varphi_q(\langle \sigma_q^E(x), \sigma_q^E(\Psi(\langle v, v \rangle) x) \rangle) h \rangle_H \\
&= \langle h, \varphi_q(\pi_q(\langle x, \Psi(\langle v, v \rangle) x \rangle)) h \rangle_H \\
&= \langle h, \varphi(\langle x, \Psi(\langle v, v \rangle) x) h \rangle_H \\
&= \langle v \otimes x \otimes h, v \otimes x \otimes h \rangle,
\end{aligned}$$

and so U_2 can be extended to a bounded linear operator U_2 from ${}_V \otimes_{\Psi} {}_E H$ onto ${}_{V_p \otimes_{\Psi_q} E_q} H$. It is easy to see that U_2 is unitary. Moreover, $U_2 \overset{V}{E} \Phi(v) = (\overset{V_p}{E_q} \Phi_q \circ \sigma_p^V) U_1(v)$ for all $v \in V$. Hence, the representations $\overset{V}{E} \Phi$ and $\overset{V_p}{E_q} \Phi_q \circ \sigma_p^V$ are unitarily equivalent. \square

Theorem 3.10. *Let $\Phi_1 : W \rightarrow B(H_1, K_1)$ and $\Phi_2 : W \rightarrow B(H_2, K_2)$ be two non-degenerate representations of W . If Φ_1 and Φ_2 are unitarily equivalent, then $\overset{V}{E} \Phi_1$ and $\overset{V}{E} \Phi_2$ are unitarily equivalent, too.*

Proof. Let $q, q' \in S(B)$, (φ_{1q}, H_1) be a representation of B_q associated to φ_1 and let $(\varphi_{2q'}, H_2)$ be a representation of $B_{q'}$ associated to φ_2 . Consider $r \in S(B)$ such that $q, q' \leq r$. By Theorem 3.9, there exists $p \in S(A)$ such that A_p acts non-degenerately on E_r and the

representation ${}^V_E\Phi_i$ is unitarily equivalent to ${}^V_{E_r}\Phi_{i_r} \circ \sigma_p^V$ for $i = 1, 2$. Since Φ_{1_r} and Φ_{2_r} are unitarily equivalent representations of W_r , Lemma 3.4 implies that the representations ${}^V_{E_r}\Phi_{1_r}$ and ${}^V_{E_r}\Phi_{2_r}$ are unitarily equivalent. \square

Corollary 3.11. *If $\Phi : W \rightarrow B(H, K)$ and $\oplus_{i \in I} \Phi_i : W \rightarrow B(\oplus_{i \in I} H_i, \oplus_{i \in I} K_i)$ are unitarily equivalent, then ${}^V_E\Phi$ and $\oplus_{i \in I} {}^V_E\Phi_i$ are unitarily equivalent.*

Proof. Let $q \in S(B)$ and $\Phi_q : W_q \rightarrow B(H, K)$ be a representation of W_q associated to Φ . For every $i \in I$, define $\Phi_{i_q} : W_q \rightarrow B(H_i, K_i)$ by $\Phi_{i_q}(\sigma_q^W(w)) = \Phi_i(w)$. If $\sigma_q^W(w) = 0$, then $\Phi_q(\sigma_q^W(w)) = 0$ and so $\Phi(w) = 0$. Since Φ and $\oplus_{i \in I} \Phi_i$ are unitarily equivalent, we conclude that $\oplus_{i \in I} \Phi_i(w) = 0$ and therefore, $\Phi_i(w) = 0$ for each $i \in I$. It proves that Φ_{i_q} is well-defined for any $i \in I$. It is easy to see that Φ_q is unitarily equivalent to $\oplus_{i \in I} \Phi_{i_q}$. By Theorem 3.9, there exists $p \in S(A)$ such that A_p acts non-degenerately on E_q and the representations ${}^V_E\Phi$ and ${}^V_{E_q}\Phi_q \circ \sigma_p^V$ of V are unitarily equivalent. The representations ${}^V_E\Phi_i$ and ${}^V_{E_q}\Phi_{i_q} \circ \sigma_p^V$, $i \in I$ are unitarily equivalent, too. On the other hand, Corollary 3.5 implies that the representations ${}^V_{E_q}\Phi_q$ and $\oplus_{i \in I} {}^V_{E_q}\Phi_{i_q}$ of V_p are unitarily equivalent. Consequently, the representations ${}^V_{E_q}\Phi_q \circ \sigma_p^V$ and $\oplus_{i \in I} ({}^V_{E_q}\Phi_{i_q} \circ \sigma_p^V)$ of V are unitarily equivalent. \square

4. THE IMPRIMITIVITY THEOREM FOR HILBERT MODULES

In this section, we introduce the concept of Morita equivalence between Hilbert modules over locally C^* -algebras and give a module version of the imprimitivity theorem.

Let A and B be locally C^* -algebras. We say that A and B are *strongly Morita equivalent*, written $A \sim_M B$, if there is a full Hilbert A module E such that locally C^* -algebras B and $K_A(E)$ are isomorphic. Joita [10, Proposition 4.4] showed that strong Morita equivalence is an equivalence relation in the set of all locally C^* -algebras. The vector space $\tilde{E} := K_A(E, A)$ is a full Hilbert $K_A(E)$ -module with the following action and inner product

$$\begin{aligned} (T, S) &\rightarrow TS, \quad S \in K_A(E), \quad T \in K_A(E, A), \\ \langle T, S \rangle &= T^*S, \quad T, S \in K_A(E, A). \end{aligned}$$

Since locally C^* -algebras B and $K_A(E)$ are isomorphic, \tilde{E} may be regarded as a Hilbert B -module. Moreover, the linear map α from A to $K_B(\tilde{E})$ defined by $\alpha(a)(\theta_{b,x}) = \theta_{ab,x}$ is an isomorphism of locally C^* -algebras by [10, Lemma 4.2 and Remark 4.3]. It is easy to see that for each $p \in S(A)$, the linear map $U_p : (\tilde{E})_p \rightarrow \tilde{E}_p$ defined by $U_p(T + N_p^{\tilde{E}}) = (\pi_p)_*(T)$ is unitary and so the Hilbert $K_{A_p}(E_p)$ -modules $(\tilde{E})_p$ and \tilde{E}_p are the same.

Definition 4.1. Suppose V and W are Hilbert modules over locally C^* -algebras A and B , respectively. The Hilbert modules V and W are called Morita equivalent if $K_A(V)$ and $K_B(W)$ are strong Morita equivalent as locally C^* -algebras. In this case, we write $V \sim_M W$.

Lemma 4.2. *Let V be a full Hilbert module over locally C^* -algebra A . Then $K_A(V)$ is strong Morita equivalent to $\overline{\langle V, V \rangle}$.*

Proof. The module $\tilde{V} = K_A(V, A)$ is a full Hilbert $K_A(V)$ -module by [10, Corollary 3.3]. Then locally C^* -algebras $K_{K_A(V)}(\tilde{V})$ and $K_A(A)$ are isomorphic by Lemma 4.2 in [10]. Since $\overline{\langle V, V \rangle} = A \simeq K_A(A)$, locally C^* -algebras $K_A(V)$ and $\overline{\langle V, V \rangle}$ are strong Morita equivalent. \square

Corollary 4.3. *Two full Hilbert modules over locally C^* -algebras are Morita equivalent if and only if their underlying locally C^* -algebras are strong Morita equivalent.*

Theorem 4.4. *Let V and W be two full Hilbert modules over locally C^* -algebras A and B , respectively, such that $V \sim_M W$. If E is a Hilbert A -module which gives the strong Morita equivalence between A and B , and Φ is a non-degenerate representation of V , then Φ is unitarily equivalent to $\tilde{V}_E^V(W_E \Phi)$.*

Proof. Let $p \in S(A)$ and Φ_p be a non-degenerate representation of V_p associated to Φ . Using [11, Lemma 4.1], there is $q \in S(B)$ such that $A_p \sim_M B_q$ and E_p gives the strong Morita equivalent between A_p and B_q . The representations φ_p and $\tilde{E}_p^{A_p(B_q)}(\varphi_p)$ of A_p are unitarily equivalent by [15, Theorem 3.29]. Then the representations Φ_p and $\tilde{E}_p^{V_p(W_q)}(\Phi_p)$ of V_p are unitarily equivalent by Lemma 3.3 and consequently, the representations $\tilde{E}_p^{V_p(W_q)}(\Phi_p) \circ \sigma_p^V$ and $\Phi_p \circ \sigma_p^V = \Phi$ of V are unitarily equivalent. In view of Theorems 3.9 and 3.10, we have

- the representations $\tilde{E}_E^W \Phi$ and $\tilde{E}_p^{W_q} \Phi_p \circ \sigma_q^W$ of W are unitarily equivalent,
- the representations $\tilde{E}_E^V(W_E \Phi)$ and $\tilde{E}_E^{(W_q \Phi_p \circ \sigma_q^W)}$ of V are unitarily equivalent, and
- the representations $\tilde{E}_E^{(W_q \Phi_p \circ \sigma_q^W)}$ and $\tilde{E}_p^{(W_q \Phi_p \circ \sigma_q^W)_q} \circ \sigma_p^V$ of V are unitarily equivalent.

The assertion now follows from the fact that $(\tilde{E}_p^{W_q} \Phi_p \circ \sigma_q^W)_q = \tilde{E}_p^{W_q} \Phi_p$. \square

We now reformulate the imprimitivity theorem within the framework of Hilbert modules as follows.

Theorem 4.5. *Let V and W be two Hilbert modules over locally C^* -algebras A and B , respectively. If $V \sim_M W$, then there is a bijective correspondence between equivalence classes of non-degenerate representations of V and W .*

Proof. By replacing the underlying C^* -algebras A and B , we may assume that V and W are full Hilbert modules over A and B , respectively. Let E be a Hilbert A -module which gives strong Morita equivalence between A and B . Then, by Theorems 3.10 and 4.4, the map $\Phi \mapsto \begin{smallmatrix} W \\ E \end{smallmatrix} \Phi$ from the set of all non-degenerate representations of V to the set of all non-degenerate representations of W induces a bijective correspondence between equivalence classes of non-degenerate representations of V and W . \square

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REFERENCES

- [1] Gh. Abbaspour Tabadkan and S. Farhangi, Induced representations of Hilbert C^* -modules, arXiv:1403.2256 [math.OA], 2014.
- [2] P. Ara, Morita equivalence and Pedersen ideals, *Proc. Amer. Math. Soc.* **129** (2001), 1041–1049.
- [3] Lj. Arambašić, Irreducible representations of Hilbert C^* -modules, *Math. Proc. R. Ir. Acad.* **105A** (2005), 11–24.
- [4] W. Beer, On Morita equivalence of nuclear C^* -algebras, *J. Pure Appl. Algebra*, **26** (1982), 249–267.
- [5] M. Fragoulopoulou, *Topological algebras with involution*, North Holland, Amsterdam, 2005.
- [6] A. Inoue, Locally C^* -algebras, *Mem. Faculty Sci. Kyushu Univ. Ser. A* **25** (1971), 197–235.
- [7] M. Joita, On Hilbert modules over locally C^* -algebras, *An. Univ. Bucuresti, Mat.* **49** (2000), 41–51.
- [8] M. Joita, *Hilbert modules over locally C^* -algebras*, University of Bucharest Press, 2006.
- [9] M. Joita, Tensor products of Hilbert modules over locally C^* -algebras, *Czech. Math. J.*, **54** (129) (2004), no. 3, 727–737.
- [10] M. Joita, Morita equivalence for locally C^* -algebras, *Bull. London Math. Soc.*, **36** (2004), no. 6, 802–810.
- [11] M. Joita, Induced representations of locally C^* -algebras, *Rocky Mountain J. Math.*, **35** (2005), no. 6, 1923–1934.
- [12] M. Joita and M. S. Moslehian, A Morita equivalence for Hilbert C^* -modules, *Stud. Math.* **209** (2012), 11–19.
- [13] G. J. Murphy, Positive definite kernels and Hilbert C^* -modules, *Proc. Edinburgh Math. Soc.* **40** (1997), 367–374.
- [14] N. C. Phillips, Inverse limit of C^* -algebras, *J. Operator Theory* **19** (1988), 159–195.
- [15] I. Raeburn and D. P. Williams, *Morita equivalence and continuous-trace C^* -algebras*, Mathematical Surveys and Monographs, 60. American Mathematical Society, Providence, RI, 1998.
- [16] M. A. Rieffel, Induced representations of C^* -algebras, *Advanced in Math.* **13** (1974), 176–257.
- [17] M. A. Rieffel, Morita equivalence for C^* -algebras and W^* -algebras, *J. Pure Appl. Alg.* **5** (1974), 51–96.
- [18] M. Skeide, Unit vectors, Morita equivalence and endomorphisms, *Publ. Res. Inst. Math. Sci.* **45** (2009), 475–518.
- [19] M. Skeide, Generalised matrix C^* -algebras and representations of Hilbert modules, *Math. Proc. R. Ir. Acad.*, **100A** (2000), 11–38.

- [20] K. Sharifi, Generic properties of module maps and characterizing inverse limits of C^* -algebras of compact operators, *Bull. Malays. Math. Sci. Soc.* **36** (2013), 481-489.
- [21] H. Zettl, Strong Morita equivalence of C^* -algebras preserves nuclearity, *Arch. Math.* **38** (1982), 448-452.

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